The Solow and basic OLG models

Laura Povoledo

Economic growth refers to an increase in a country's ability to produce goods and services. The advantage of economic growth is that an increase in real national income allows more goods for consumption.

To understand economic growth, we use the framework of analysis developed by Robert Solow, from MIT, in the late 1950s. This framework was developed to answer the following questions:

What determines growth?

What is the role of capital accumulation?

What is the role of technological progress?

1 The Aggregate Production Function

A key factor that determines the quantity of goods and services an economy can produce is the quantity of inputs that producers in the economy use.

The aggregate production function is a specification of the relation between aggregate output and the inputs in production.

$$Y = F(K, N)$$

Y = aggregate output.

K = capital — the sum of all the machines, plants, and buildings used in productive activities in the economy.

N = labour — the number of workers in the economy.

The function F, tells us how much output is produced for given quantities of capital and labour.

Example 1 $Y = K^{0.3}N^{0.7}$

If $K = 100$ *units and* $N = 500$ *units then total output produced by the economy is equal to:*

$Y = (100)^{0.3}(500)^{0.7} = 3.98 \cdot 77.5 = 308.52$ *units*

Remark 1 *The Cobb-Douglas production function is defined as follows:*

$Y = K^{\alpha}N^{\beta}$

However, it is not only the quantities of factors that are utilised that determine the amount of output produced. Equally important is how effectively these factors are used. Thus, the aggregate production function depends on the state of technology. The higher the state of technology, the higher is Y for a given K and a given N.

2 Returns to Scale and Returns to Factors

Constant returns to scale is a property of the production function such as, if the scale of operation is doubled — that is, if the quantities of capital and labour are doubled — then output will also double:

$$2Y = F\left(2K, 2N\right)$$

Or more generally, for any number x,

$$xY = F\left(xK, xN\right)$$

Example 2 *The production function $Y = K^{0.3}N^{0.7}$ is characterised by constant returns to scale:*

$$\left(xK\right)^{0.3}\left(xN\right)^{0.7} = x^{0.3}x^{0.7}\cdot K^{0.3}N^{0.7} = x\cdot K^{0.3}N^{0.7} = x\cdot Y$$

Remark 2 *The Cobb-Douglas production function $Y = K^{\alpha}N^{\beta}$ is characterised by:*

- *constant returns to scale if $\alpha+\beta=1$*
- *increasing returns to scale if $\alpha+\beta>1$*
- *decreasing returns to scale if $\alpha+\beta<1$*

Returns to scale must not be confused with returns to factors (capital and labour). When we talk about "returns to scale" we want to describe what happens to output when all inputs are changed in the same proportion. On the other hand, when we talk about "returns to a factor" we want to describe what happens to output when only one input is changed and all other inputs are held fixed. For example:

Decreasing returns to capital refers to the property that increases in capital lead to smaller and smaller increases in output as the level of capital increases.

Decreasing returns to labour refers to the property that increases in labour, given capital, lead to smaller and smaller increases in output as the level of labour increases.

Notice that it is generally true that increases in capital lead to smaller and smaller increases in output as the level of capital increases. The same is true for labour. In other words, the production function is characterised by decreasing returns to capital and by decreasing returns to labour. This happens because workers cannot be fully productive if there is not enough capital, and capital cannot be fully productive if there are not enough workers operating the machinery. So if we increase only one input (labour or capital) while the other input stays fixed, the marginal product becomes smaller and smaller.

Revision questions

1. Explain each of the following concepts:
   a. production function
   b. constant returns to scale
   c. decreasing returns to capital
   d. decreasing returns to labour
2. Which of the following is a Cobb-Douglas production function?
   a. $Y = K^{0.2}N^{0.8}$
   b. $Y = K^{0.2} + 10 \cdot N^{0.8}$
3. Determine whether each one of the following functions is characterised by increasing, decreasing or constant returns to scale:
   a. $Y = K^{0.2}N^{0.8}$
   b. $Y = K^{0.7}N^{0.8}$
   c. $Y = K^{0.2}N^{0.7}$

3 Output per Worker and Capital per Worker

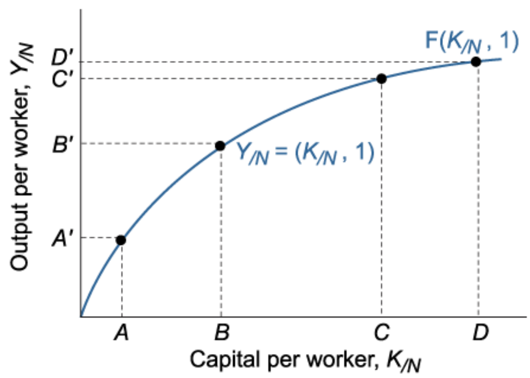
Constant returns to scale implies that we can rewrite the aggregate production function as:

$$\frac{Y}{N} = F\left(\frac{K}{N}, \frac{N}{N}\right) = F\left(\frac{K}{N}, 1\right)$$

(Why? The above equation is obtained from $xY = F(xK, xN)$, by substituting x with $\frac{1}{N}$).

The amount of output per worker, Y/N depends on the amount of capital per worker, K/N. As capital per worker increases, so does output per worker. We can illustrate this with a graph:

1.

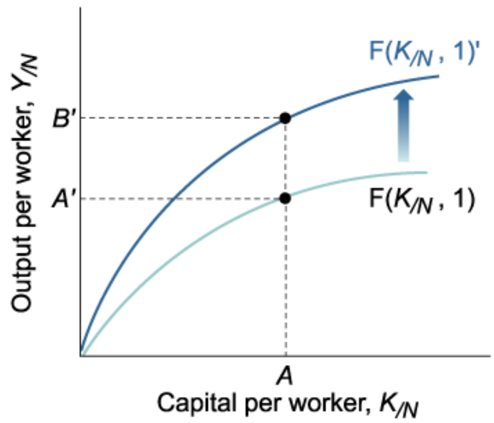


The decreasing returns assumption implies that the increase in output per worker from an extra unit of capital per worker will decline as K/N increases. Thus, the aggregate production function has the shape depicted in the figure above.

4 The Sources of Growth

An improvement in the state of technology shifts the production function up, leading to an increase in output per worker for a given level of capital per worker.

2.



The aggregate production function is:

$$\frac{Y}{N} = F\left(\frac{K}{N}, 1\right)$$

Using the above equation, we can now determine where growth comes from:

- Increases in output per worker (Y/N) can come from increases in capital per worker (K/N).
- Or they can come from improvements in the state of technology that shift the production function, F, and lead to more output per worker (Y/N) given capital per worker (K/N).

In other words, growth comes from capital accumulation and from technological progress. But capital accumulation cannot sustain growth indefinitely, because decreasing returns imply that larger and larger increases in capital per worker would be required. At some point, society would be unwilling to save the necessary resources to provide for the required increases in capital per worker, and growth would cease.

Sustained growth requires sustained technological progress. The rate of growth of output per capita is eventually determined by the economy's rate of technological progress.

Revision questions

1. You are given the following production function:

$$Y = K^{0.3}N^{0.7}$$

   a. Transform it as a relationship between output per worker and capital per worker
   b. Illustrate the production function by means of a diagram, with capital per worker K/N on the horizontal axis, and output per worker Y/N on the vertical axis.

2. Graphically illustrate and explain the effects on output per worker Y/N of:
   a. An increase in capital per worker K/N
   b. A decrease in capital per worker K/N
   c. An increase in the level of technology
   d. A decrease in the level of technology

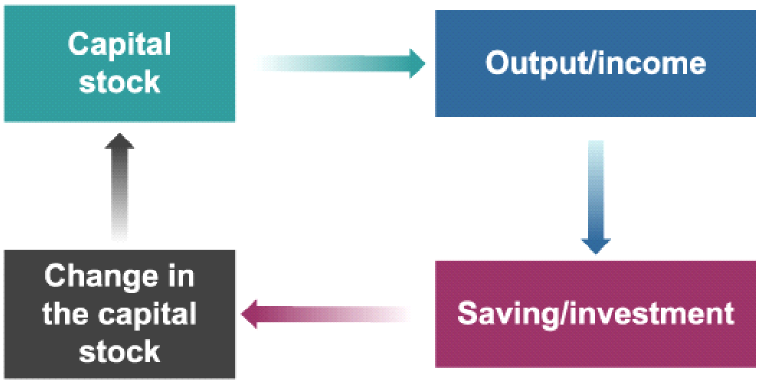
3. Which of the following will likely cause an increase in output per worker? Explain
   a. an increase in on-the-job training
   b. an increase in education expenditures
   c. an increase in capital per worker
   d. a technological improvement
   e. all of the above

5 The Effects of Capital on Output

At the center of the determination of output in the long run are two relations between output and capital:

- The amount of capital determines the amount of output being produced.
- The amount of output determines the amount of saving and investment, and so the amount of capital being accumulated.

3.



To determine the effect of capital on output, we only need three very plausible assumptions:

1. The fraction of real GDP devoted to saving (the saving rate) is constant, i.e.:

$$S = s \cdot Y \tag{1}$$

   S = aggregate savings.
   s = saving rate, i.e. proportion of output (constant) that is saved each year. This assumption captures the empirical regularity that the saving rate (s) does not appear to change systematically as a country increases its income and that savings rates in rich countries do not appear to differ systematically from savings rates in poor countries.
2. The economy is closed, so aggregate domestic savings are equal to aggregate investment:

$$I = S \tag{2}$$

3. Capital depreciates at the constant rate δ. Thus, the change in the capital stock over time is:

$$K_{t+1} = (1 - \delta)\, K_t + I_t \tag{3}$$

   Thus, investment creates new capital, but the existing capital stock depreciates over time.[1]

Combining equation (1) with equation (2), we obtain:

$$I_t = s \cdot Y_t$$

Therefore we can show that output affects capital accumulation over time:

$$K_{t+1} = (1 - \delta) K_t + s \cdot Y_t$$

6 Technological Progress and the Production Function

Technological progress has many dimensions. It may mean:

- Larger quantities of output
- Better products
- New products
- A larger variety of products

Technological progress leads to increases in output for given amounts of capital and labour. Therefore, we can say that technological progress is anything that increases the efficiency of either labour or capital, or both.

Let the variable A denote the state of technology. Then write the aggregate production function as:

$$Y = F(K, N, A)$$

The equation above shows that output depends on both capital and labour (K and N), and on the state of technology (A). A more restrictive but more convenient form is:

$$Y = F(K, A \cdot N)$$

We can describe $A \cdot N$ as the amount of effective labour, or labour in "efficiency units" in the economy. For example, take any 2 countries, named B and C, endowed with the same amount of capital and the same number of workers (for example, there are $N = 40$ million workers in both countries). Assume now that country B is technologically more advanced than country C, so that country B's workers are twice as productive than country C's workers. Therefore, we can say that country B has 80 million units of effective labour, and country C has 40 million units of effective labour. This way of describing the state of technology is very convenient. It implies that country B is able to produce the same amount of output than country C, with only half of the labour input. In practice, technological progress reduces the number of workers needed to achieve a given amount of output.

With constant returns to scale,

---

[1] Notice that if $I_t = \delta K_t$, then the capital stock is constant over time:

$$K_{t+1} = (1 - \delta) K_t + \delta K_t = K_t$$

So, each year an amount of investment equal to δK_t is needed just to keep the capital stock constant.

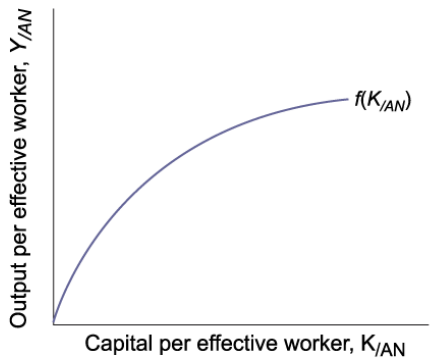
$$xY = F\left(xK, xAN\right)$$
so if we substitute x with $\frac{1}{AN}$ we obtain:

$$\frac{Y}{AN} = F\left(\frac{K}{AN}, \frac{AN}{AN}\right) = F\left(\frac{K}{AN}, 1\right)$$
which we can redefine as:

$$\frac{Y}{AN} = f\left(\frac{K}{AN}\right)$$

In words this means that *output per effective worker* is a function of *capital per effective worker*. Notice that *higher* capital per effective worker leads to *higher* output per effective worker. We can illustrate this production function with a graph:

4.



7 Interactions Between Output and Capital

How do output and capital per effective worker, Y/AN and K/AN , evolve over time? To answer these questions, we will make use of some fundamental relationships.

- The relation between output per (effective) worker and investment per (effective) worker

  $$I = S = s \cdot Y$$
  Dividing both sides by AN, we get

  $$\frac{I}{AN} = s \cdot \frac{Y}{AN}$$
  Notice that, because $\frac{Y}{AN} = f\left(\frac{K}{AN}\right)$, then:

  $$\frac{I}{AN} = s \cdot f\left(\frac{K}{AN}\right)$$
- The relation between investment and capital accumulation (3):

$$K_{t+1} = (1-\delta) K_t + I_t$$

Let's now ask what investment has to be just to maintain a given level of capital per effective worker, K/AN. The answer depends on two factors: the growth of AN , and the rate of depreciation of the existing capital stock.

1. The growth of AN

Assume that technology is growing at the rate and g_A :

$$g_A \equiv \frac{\Delta A}{A} = \frac{A_{t+1} - A_t}{A_t}$$

Assume also that population is growing at rate g_N . If we assume that the ratio of workers to the total population remains constant, the number of workers, N, also grows at rate g_N :

$$g_N \equiv \frac{\Delta N}{N} = \frac{N_{t+1} - N_t}{N_t}$$

Together, these two assumptions imply that the growth rate of effective labour AN is equal to $g_A + g_N$:

$$\frac{\Delta (AN)}{AN} = \frac{A_{t+1}N_{t+1} - A_t N_t}{A_t N_t} = g_A + g_N$$

Example: if the number of workers increases by 1% per year and the rate of technological progress is 2%, then the growth rate of effective labour is equal to 3%.

In this model, in order for $\frac{K}{AN}$ to be constant, K must grow at the same rate as NA, so in the steady state:

$$\frac{K}{AN} \text{ is constant if and only if } \frac{\Delta K}{K} = g_A + g_N$$

Example: if effective labour increases by 3%, then the capital stock must increase by 3% per year so that the ratio $\frac{K}{AN}$ stays constant over time. Notice that this also implies:

$$\frac{K}{AN} \text{ is constant if and only if } \Delta K = (g_A + g_N) K$$

2. The rate of depreciation of the existing capital stock

What is the amount of investment per effective worker needed to keep a constant ratio $\frac{K}{AN}$?

Consider equation (3) again:

$$K_{t+1} = (1-\delta) K_t + I_t$$

$$\Delta K = -\delta K + I$$

Now, we know that a necessary requirement for $\frac{K}{AN}$ to be constant is that ΔK must be equal to $(g_A + g_N)\,K$. If we substitute this requirement in the equation above we obtain:

$$(g_A + g_N)\,K = -\delta K + I$$

which implies:

$$I = (g_A + g_N + \delta)\,K \tag{4}$$

Therefore, if $I = (g_A + g_N + \delta)\,K$, then $\frac{\Delta K}{K} = g_A + g_N$, which implies that $\frac{K}{AN}$ is constant over time. Now divide both the left and the right-hand side by AN:

$$\frac{I}{AN} = (g_A + g_N + \delta)\,\frac{K}{AN}$$

We are now ready to state our first result: the amount of investment per effective worker needed to maintain a constant level of capital per effective worker is $(g_A + g_N + \delta)\,\frac{K}{AN}$. This is the amount of investment per effective worker that is sufficient both to replace depreciated capital and to allow the capital stock to increase by $(g_A + g_N)\,K$.

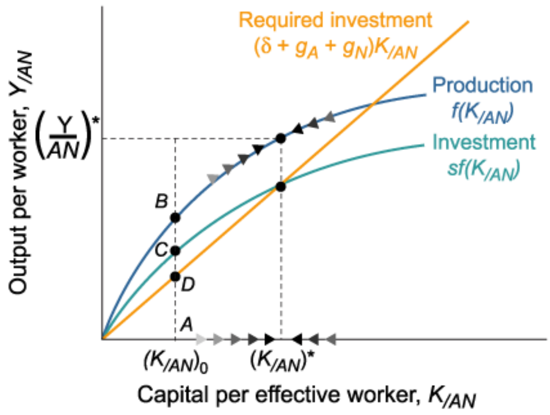
If investment per effective worker exceeds $(g_A + g_N + \delta)\,\frac{K}{AN}$, the change in capital per effective worker is positive: $\frac{K}{AN}$ increases.

If investment per effective worker is less than $(g_A + g_N + \delta)\,\frac{K}{AN}$, the change in capital per effective worker is negative: $\frac{K}{AN}$ decreases.

8 Dynamics of Capital and Output

We can now give a graphical description of the dynamics of capital per effective worker and output per effective worker. The following figure focuses on output, capital, and investment per effective worker, rather than per worker:

5.



The relation between investment per effective worker and capital per effective worker is drawn as the upper curve, multiplied by the saving rate, s.

Consider an initial, given level of capital per effective worker, called $\left(\frac{K}{AN}\right)_0$. At that level, output per effective worker is given by the distance AB. Investment per effective worker is given by AC. The amount of investment (per effective worker) required to maintain that level of capital per effective worker is equal to the distance AD. Since AD is less than AC, the economy is investing more than what required to keep $\frac{K}{AN}$ constant. As a result, capital per effective worker increases over time.

The process continues until the economy reaches $\left(\frac{K}{AN}\right)^*$. At this point, you can see from the diagram that the amount of investment in the economy is exactly equal to what is required to keep a constant level of capital per effective worker. Therefore, once the economy reaches $\left(\frac{K}{AN}\right)^*$, it will stay there.

Hence, $\left(\frac{K}{AN}\right)^*$ is called the *steady state* of the economy. It is called like that because it is a stable equilibrium: if the economy is there, it will never leave and if the economy is not there, it tends to converge to it.

The steady state can be determined by solving the following equation:

$$s \cdot f\left(\frac{K}{AN}\right)^* = (g_A + g_N + \delta)\left(\frac{K}{AN}\right)^*$$

Notice that, once we know $\left(\frac{K}{AN}\right)^*$, we can determine everything else:

- steady state output per effective worker:

$$\left(\frac{Y}{AN}\right)^* = f\left(\frac{K}{AN}\right)^*$$

- steady state investment per effective worker:

$$\left(\frac{I}{AN}\right)^* = s\cdot\left(\frac{Y}{AN}\right)^* = s\cdot f\left(\frac{K}{AN}\right)^*$$

Notice that, in the steady state, what is constant is $\frac{Y}{AN}$, not output itself. In the steady state, output Y grows at the same rate as effective labor AN; effective labor grows at a rate $(g_A + g_N)$; therefore, output growth in steady state equals $(g_A + g_N)$.

Capital per effective worker also grows at a rate equal to $(g_A + g_N)$.

The growth rate of output is independent of the saving rate.

Because output, capital, and effective labor all grow at the same rate, $(g_A + g_N)$, the steady state of the economy is also called a state of *balanced growth*.

In the steady state (equivalently, on a balanced growth path; equivalently, in the long run):

- Capital per effective worker and output per effective worker are constant.
- Capital per worker and output per worker are growing at the rate of technological progress, g_A.
- Labor is growing at the rate of population growth, g_N; capital and output are growing at a rate equal to the sum of population growth and the rate of technological progress, $(g_A + g_N)$.

6.

Table 12-1 The Characteristics of Balanced Growth

		Rate of growth of:
1	Capital per effective worker	0
2	Output per effective worker	0
3	Capital per worker	g_A
4	Output per worker	g_A
5	Labor	g_N
6	Capital	$g_A + g_N$
7	Output	$g_A + g_N$

7.

Revision questions

1. Explain what factors determine how much investment is required to maintain a given level of capital per effective worker.

2. Suppose the economy's production function is $Y = \sqrt{K}\sqrt{AN}$ and both the saving rate (s) and the depreciation rate (δ) are equal to 0.10. Further, suppose that the number of workers grows at 1.5% per year (g_N = 0.015) and the rate of technological progress is 3.5% per year (g_A = 0.035).

   a. Write the production function in units of effective labour, and draw it in a graph. Add to the graph a line representing the amount of investment that is required to keep K/AN constant over time.

   b. Find the steady-state values of:

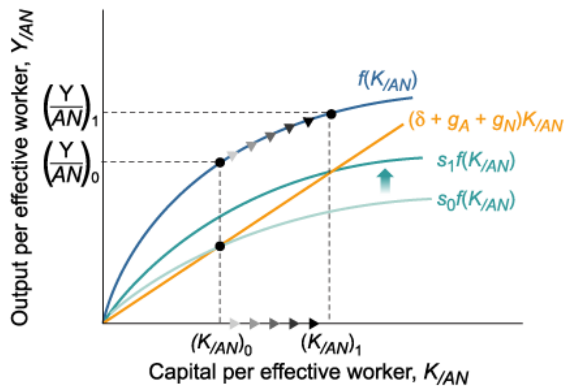
      i The capital stock per effective worker

      ii Output per effective worker

3. Assume the economy has achieved the balanced growth steady state. Explain what factors determine the rates of growth of each of the following variables when balanced growth is achieved: output per effective worker, capital per effective worker, output per worker, capital per worker, output, and capital.

9 The Effects of an Increase in the Saving Rate

8.



An increase in the saving rate leads to an increase in the steady-state levels of output per effective worker and capital per effective worker.

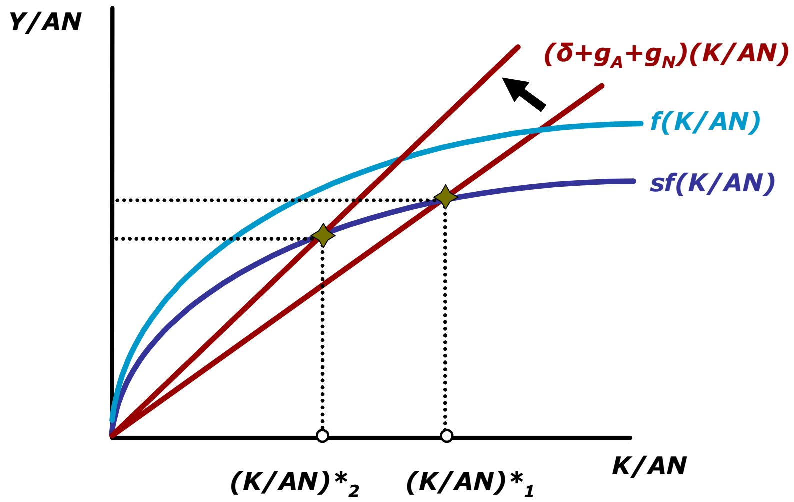
Note that, in the transition from the old to the new steady state, the economy experiences a period of "accelerated growth" = since both K/AN and Y/AN are growing, K/N and Y/N must be growing faster than g_A , and K and Y must be growing faster than $(g_A + g_N)$.

But in the new steady state, we will have again "balanced growth", since both K/AN and Y/AN are constant, K/N and Y/N grow at the rate of technological progress, g_A , etc.

An increase in the savings rate has a temporary rising effect on growth rates but no permanent effects. It has however a permanent rising effect on levels of capital and output per worker. "Level effect but no growth effect".

10 The effects of an increase in the rate of technological progress or population growth

9.



Suppose that either the rate of technological progress g_A, or the population growth rate g_N, grows. This shifts the line $(g_A + g_N + \delta)\,\frac{K}{AN}$: the amount of investment (per effective worker) needed to maintain a constant level of capital (per effective worker) increases.

The economy moves to a new steady state with lower income per capita $\frac{Y}{AN}$.

11 The Solow model and the central questions of economic growth

Fast growth in output Y may come from two sources:

- A higher rate of technological progress. If g_A is higher, balanced output growth $(g_Y = g_A + g_N)$ will also be higher. In this case, the rate of growth of output per capita $\frac{Y}{N}$ equals the rate of technological progress g_A .
- Adjustment of capital per effective worker, $\frac{K}{AN}$, to a higher level. In this case, the rate of growth of output per capita $\frac{Y}{N}$ temporarily exceeds the rate of technological progress g_A.

However, only a higher rate of technological progress can lead to a permanent increase in the growth rate of output per worker $\frac{Y}{N}$. As a result, only differences in technology have any reasonable hope of accounting for the vast differences in wealth across time and space that can be observed in the real world. The mere accumulation of capital is not the explanation for the difference in the growth performance of countries around the world.

Revision questions

1. Graphically illustrate and explain the effects in the Solow growth model of:
   a. an increase in the saving rate;
   b. an increase in the rate of technological progress;
   c. an increase in population growth.

   In your answer, you must clearly label all curves and explain what happens to the rate of growth of output in the new steady state.

12 Empirical applications of the Solow growth model

Empirical work on the Solow growth model usually proceeds from the assumption that, in principle, the same production technologies must be available to all countries. This means that what is usually assumed is that both the production function $F(K, AN)$, the absolute level of technology A and the rate of technological progress g_A are the same across countries.

If this assumption is true, differences in per capita incomes can be explained in two ways (not mutually exclusive):

1. Countries may have different steady states. This is because countries differ in their savings rate s, depreciation rate δ or rate of population growth g_N . All of them affect the steady state of the model:

$$s \cdot f\left(\frac{K}{AN}\right)^{*} = (g_A + g_N + \delta)\left(\frac{K}{AN}\right)^{*}$$

2. Countries may not have yet settled into their steady states, adjustment may be slow so most countries are still on a transition path. So incomes across the world may differ, even if all countries had the same steady state.

Here we will focus on explanation 2. In this case, the Solow model yields an interesting proposition regarding the relationship between the *level* of per capita income $\frac{Y}{N}$ and the *growth rate* of $\frac{Y}{N}$.

Per capita income in countries that are in the steady state only grows at the rate of technological progress g_A . If a country's income is below the steady state, the growth rate of $\frac{Y}{N}$ will be higher than g_A . If a country's income is above the steady state, the growth rate of $\frac{Y}{N}$ will be lower than g_A . All this can be generalised into the so-called *absolute convergence hypothesis*, which states that there is a negative relationship between a country's initial level of income per capita and the growth rate. The absolute convergence hypothesis is the direct prediction of the Solow model. The Solow model predicts that, other things equal, poor countries (with lower Y/N and K/N) should grow faster than rich ones. If true, then the income gap between rich and poor countries would shrink over time, and living standards converge.

A way to test the absolute convergence hypothesis is by regressing the growth rate of output per capita on a constant and the initial level of output per capita for a sample of countries:

$$\ln\left(\frac{Y}{N}\right)_{i,T} - \ln\left(\frac{Y}{N}\right)_{i,0} = \alpha + \beta \cdot \ln\left(\frac{Y}{N}\right)_{i,0} + \varepsilon_i$$

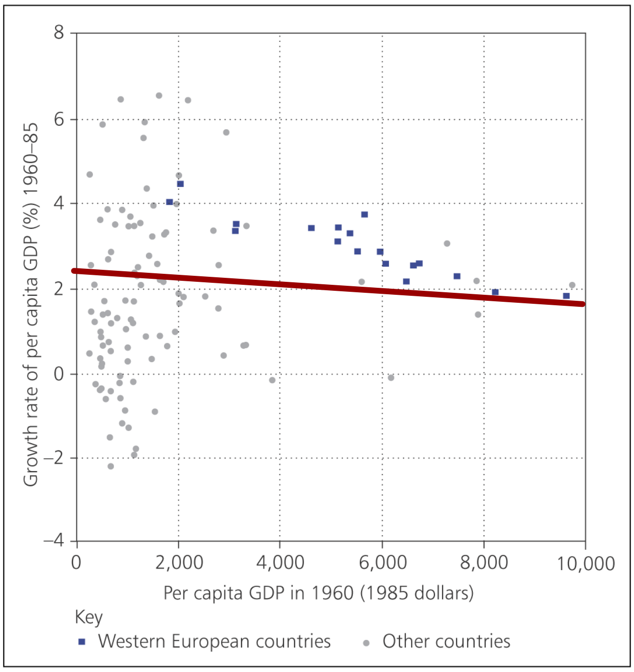
where $\ln(Y/N)$ is the log of output per person, T is the last year in the sample, 0 denotes the earliest year in the sample, ε is an error term, and i indexes countries. If there is convergence, β will be negative: a higher initial output is associated with lower growth, in accordance with the absolute convergence hypothesis. If the estimated β is positive or equal to zero, then the absolute convergence hypothesis is not valid in the data (a value for β of 0 would imply that the growth rate is uncorrelated with initial output, and a negative value for β would imply that countries with higher output grow faster).

What are the results from the empirical research?

Baumol (1986) examines convergence from 1870 to 1979 among 16 industrialised countries. According to his estimates, $\widehat{\beta} = -0.995$. This result is in accordance with the absolute convergence hypothesis, since it suggests almost perfect convergence.

However, other studies which included a larger set of countries typically found no convergence in the data. The figure below reports the scatter plot and the estimated regression line for 122 countries.

10.



There are two messages in this data plot. First, there is no worldwide convergence of output (the coefficient of the regression line is almost zero). Many poor countries grow more slowly than the rich countries, thus widening the income gap. Second, within relatively homogeneous groups of countries (the Western European countries that have been singled out in the scatter plot), convergence does indeed occur.

Do these two observations contradict the Solow growth model? No, since the Solow model only proposes *absolute convergence* for countries with the same steady states (such as Western European countries). But across continents, sizeable differences in the steady states exist and the Solow model would only postulate convergence to those specific steady states. This is called the *relative convergence hypothesis*.

Notice that:

Absolute convergence means convergence to the same steady-state value $\frac{Y}{N}$ across countries. Relative convergence allows for differences in the steady-state value $\frac{Y}{N}$ across countries (due, for example, to differences in the savings rate).

Both absolute and relative convergence state that there should be convergence to the same growth rate of output per capita $\frac{Y}{N}$.

Revision questions

1. How would you test the Solow growth model?
2. Discuss the results from the empirical research on the Solow growth model.
3. Illustrate both the absolute and the relative convergence hypothesis. Under the relative convergence hypothesis, the coefficient $\widehat{\beta}$ is negative only if estimated among countries that have similar technological possibilities, savings rates, population growth rates, etc. Why?

Appendix 1: Logarithms

The logarithm is the mathematical operation that is the inverse of exponentiation (raising a constant to a power). The logarithm of a number X in base b is the number n such that $X = b^n$. It is usually written as:

$$\log_b(X) = n$$

For example, $\log_3(81) = 4$, since $3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$.

The logarithm in base e (the Euler's number $e = 2.718...$) is called the natural logarithm and it is denoted with $\ln(X)$. The logarithm in base 10 is called the common logarithm and it is denoted with $\log(X)$.

Logarithms possess some properties that assist with model-building and the visual display of models in graphs. Since we are not interested in the higher mathematics of logarithms, we skip proofs and illustrate the concepts needed by means of numerical examples.

Linearization property

The $\log$ function has the defining property that

$$\log(XY) = \log(X) + \log(Y)$$

i.e., the logarithm of a product equals the sum of the logarithms. Therefore, logging tends to convert multiplicative relationships to additive relationships.

Growth rates

As a rule of thumb, growth rates may be approximated by the change in the logarithm of the variable under consideration:

$$\frac{X_{t+1} - X_t}{X_t} \simeq \log(X_{t+1}) - \log(X_t)$$

Appendix 2: The simple regression model

Definition: it is a linear model that relates two variables, X and Y. It can be written as:

$$Y = \alpha + \beta \cdot X + \varepsilon$$

Y may be referred to as "dependent variable" or "explained variable".

X may be referred to as "independent variable" or "explanatory variable".

ε is referred to as the error term or disturbance. This is because the relationship between X and Y is never going to be perfect (there may be factors other than X that affect Y). However we can assume that the errors on average cancel each other out, i.e. have a 0-mean.

α is the intercept.– it gives the value of Y when $X = 0$ and $\varepsilon = 0$.

β is the slope.– it relates a change in Y to a change in X (holding ε constant).

Suppose we have a sample of $i = 1, ...N$ observations. Our goal is to get reliable estimators of α and β. Here we will restrict ourselves to one method of estimation, the Ordinary Least Squares or OLS method. This method computes the estimated values of α and β by minimising the sum of squared errors:

$$\min_{\widehat{\alpha},\widehat{\beta}} \sum_{i=1}^{N} (\widehat{\varepsilon}_i)^2 = \sum_{i=1}^{N} \left(Y_i - \widehat{\alpha} - \widehat{\beta} \cdot X_i\right)^2$$

Notice that the estimated parameters are denoted with a "hat". The estimated value of Y when using estimates of the regression coefficients is given by:

$$\widehat{Y}_i = \widehat{\alpha} + \widehat{\beta} \cdot X_i$$

and the estimated values of Y are equal to the actual values minus the estimated errors:

$$\widehat{Y}_i = Y_i - \widehat{\varepsilon}_i$$

The OLS method is routinely implemented in all econometrics and statistical packages: Stata, EViews, ... We skip the proofs and we report here only the formulas used in the computation of $\widehat{\alpha}$ and $\widehat{\beta}$.

Denote with $\overline{X}$ and $\overline{Y}$ the expected values (averages) of X and Y. Then it can be shown that:

$$\widehat{\beta} = \frac{\sum_{i=1}^{N} \left(Y_i - \overline{Y}\right)\left(X_i - \overline{X}\right)}{\sum_{i=1}^{N} \left(X_i - \overline{X}\right)^2} = \frac{cov\,(X, Y)}{var\,(X)}$$

$$\widehat{\alpha} = \overline{Y} - \widehat{\beta} \cdot \overline{X}$$

In the Solow model, the saving rate is considered constant. How important is this assumption? How would the results change if saving/investment was treated as the result of choices made at the household/firm level? We can answer this question by means of the Diamond model.

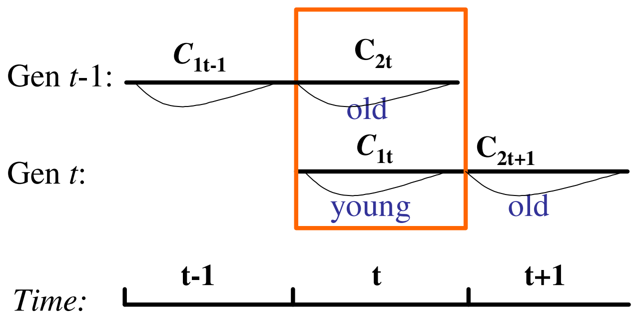
13 The Diamond model: Assumptions

N_t individuals are born in period t. Each individual/consumer lives for two periods only. Population (or labour force) grows at rate g_N:

$$N_t = (1 + g_N)\, N_{t-1}$$

Each individual supplies one unit of labour in the first period of her life. In the first period the individual divides the labour income between consumption and saving. In the second period the individual simply consumes her savings and any earned interest.

11.



The utility of an individual born at t, denoted U_t, is given by:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

- The parameter $\theta > 0$ is called the "risk aversion" parameter. It determines the willingness of a household to shift first-period consumption to the second period. The smaller is θ, the more willing is the household to shift consumption between the two periods. The higher is θ, the more risk-averse the household is which means she prefers to consume current income in the current period and would ask for a high return on savings before she shifts a small fraction of income.
- The parameter ρ is called the "time discount" rate. Since $\rho > 0$, the household places a greater weight on first period consumption than on second-period consumption.

Production is characterised by the same assumptions as before: constant returns to scale, exogenous constant growth rate of labor augmenting technical change:

$$Y_t = F\left(K_t, A_t N_t\right)$$

$$A_t = \left(1 + g_A\right) A_{t-1}$$

The intensive-form of the production function is:

$$y_t = f\left(k_t\right)$$

where $y_t \equiv \frac{Y_t}{A_t N_t}$ and $k_t \equiv \frac{K_t}{A_t N_t}$. Markets are assumed to be perfect, i.e. every input earns its marginal product and firms make zero profits:

- the real interest rate equals the marginal product of capital: $r_t = f'\left(k_t\right)$
- the wage per unit of effective labor is given by: $w_t = f\left(k_t\right) - k_t f'\left(k_t\right)$

In each period, the capital owned by the old and the labor supplied by the young combine to produce output that is distributed according to marginal products. We assume also that there is no depreciation, $\delta = 0$.

14 Household behaviour

The young choose how much to consume of labor income and how much to save by maximising utility subject to the budget constraint:

$$\max \quad \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

$$s.t. \quad C_{1t} + \frac{C_{2t+1}}{1+r_{t+1}} = A_t w_t$$

The first-order condition gives:

$$\frac{C_{2t+1}}{C_{1t}} = \left[\frac{1+r_{t+1}}{1+\rho}\right]^{1/\theta}$$

Notice that whether lifetime consumption profiles are rising or falling depends on whether the real rate of return r_{t+1} is greater or less than the rate of time discount ρ.

We can use the first-order condition and the budget constraint to express C_{1t} as a function of income and the interest rate. In this way we will also be able to express savings as a function of income and the interest rate:

$$
\begin{aligned}
C_{1t} + \frac{1}{1+r_{t+1}}\left[\frac{1+r_{t+1}}{1+\rho}\right]^{1/\theta} C_{1t} &= A_t w_t \\
C_{1t} + \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} C_{1t} &= A_t w_t \\
C_{1t}\left[1 + \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right] &= A_t w_t \\
C_{1t}\left[\frac{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right] &= A_t w_t \\
C_{1t} &= \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} A_t w_t
\end{aligned}
$$

So C_{1t} is a function of tastes (ρ and θ), technology A_t and market prices (wages today w_t and rate of return of capital during retirement r_{t+1}). Letting s denote the fraction of income saved:

$$
\begin{aligned}
s(r) &= 1 - \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} \\
s(r) &= \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} \qquad (5)
\end{aligned}
$$

It is possible to prove that the young individuals' saving is increasing in r_{t+1} if and only if $\theta > 1$.

15 The dynamics of the economy

The capital stock in period t+1 is the amount saved by young individuals in period t. Thus:

$$K_{t+1} = s\left(r_{t+1}\right) A_t N_t w_t$$

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1}N_{t+1}} = s\left(r_{t+1}\right) \frac{A_t N_t}{A_{t+1}N_{t+1}} w_t$$

Since $A_{t+1}N_{t+1} = \left(1+g_A\right)\left(1+g_N\right) A_t N_t$:

$$k_{t+1} = \frac{K_{t+1}}{A_{t+1}N_{t+1}} = \frac{1}{\left(1+g_A\right)\left(1+g_N\right)} s\left(r_{t+1}\right) w_t$$

Now substitute in $r_{t+1} = f'\left(k_{t+1}\right)$ and $w_t = f\left(k_t\right) - k_t f'\left(k_t\right)$:

$$k_{t+1} = \frac{1}{\left(1+g_A\right)\left(1+g_N\right)} s\left(f'\left(k_{t+1}\right)\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right] \quad (6)$$

Equation (6) shows that k_{t+1} is a function of k_t. A value of k_t such that $k_{t+1} = k_t$ satisfies (6) is an equilibrium (or steady state) value of k.

Easy case: Logarithmic utility ($\theta = 1$) and Cobb-Douglas production

If $\theta = 1$ then:

$$s\left(r\right) = \frac{1}{\left(1+\rho\right)+1} = \frac{1}{2+\rho}$$

If $Y_t = K_t^{\alpha}\left(A_t N_t\right)^{1-\alpha}$ then:

$$f\left(k_t\right) = f\left(\frac{K_t}{A_t N_t}\right) = \left(\frac{K_t}{A_t N_t}\right)^{\alpha} = k_t^{\alpha}$$

$$f'\left(k_t\right) = \alpha k_t^{\alpha-1}$$

$$w_t = f\left(k_t\right) - k_t f'\left(k_t\right) = k_t^{\alpha} - k_t \cdot \alpha k_t^{\alpha-1} = \left(1-\alpha\right) k_t^{\alpha}$$

Equation (6) becomes:

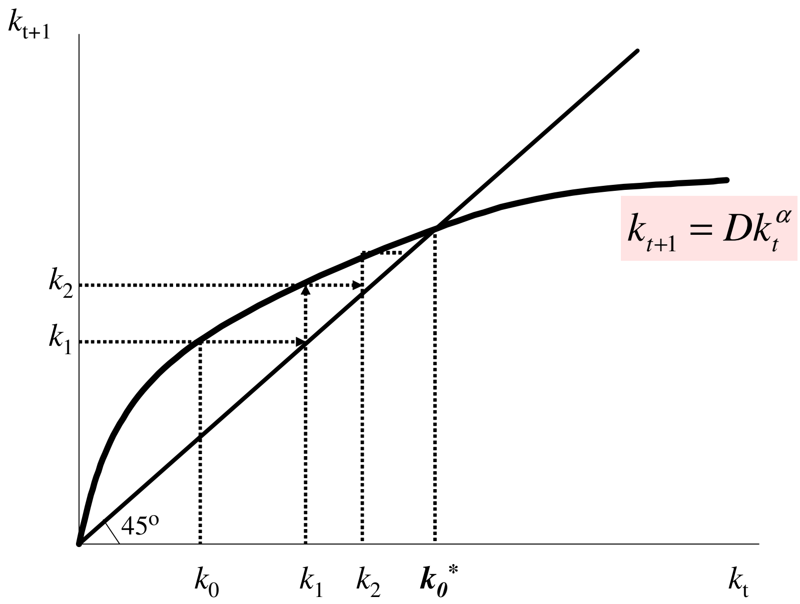
$$k_{t+1} = \frac{1}{\left(1+g_A\right)\left(1+g_N\right)} \frac{1}{2+\rho} \left(1-\alpha\right) k_t^{\alpha}$$

The following figure shows k_{t+1} is a function of k_t. The point where the function intersects the 45-degree line is the equilibrium or steady state of the model.

In the case of logarithmic utility and Cobb-Douglas production there is only one equilibrium level of capital (aside from $k = 0$) such that $k_{t+1} = k_t$. It is given by:

$$k^* = \left[\frac{1-\alpha}{\left(1+g_A\right)\left(1+g_N\right)\left(2+\rho\right)}\right]^{\frac{1}{1-\alpha}}$$

k^* is globally stable: wherever k starts (other than 0) it converges to k^*.



How does the economy adjust to the steady state? Starting with a low initial capital ($k_0 < k^*$), the savings of the young generation gradually increase the capital stock. Each generation is better off than the previous one because the increase in the capital stock done by the saving of the old raises wages for next generation's young people.

What happens once we reach the k^*?

Similar to the Solow model: in this case,

- output and capital per effective labor stays constant
- output and capital per worker grow at rate g_A
- output and capital would increase at that rate $g_A + g_N$

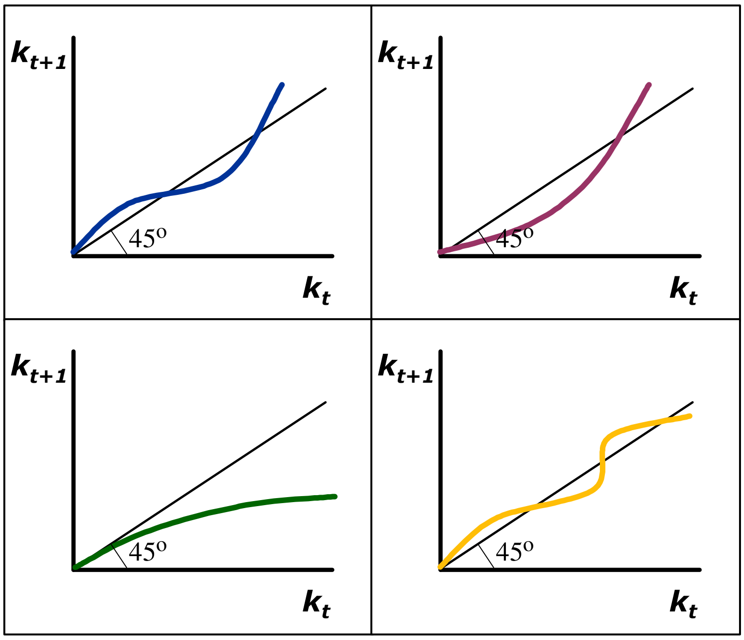
So, again, technological progress is the only engine of long-run growth in this model.

The general case

The following figures shows some possible forms for the relationship between k_{t+1} and k_t :

We may have one, multiple or no steady state, as the figures illustrate. Thus, the Diamond model has potentially important implications for the dynamics of the

13.



economy: a stable equilibrium may not be possible, or it may depend on initial conditions.

Revision questions

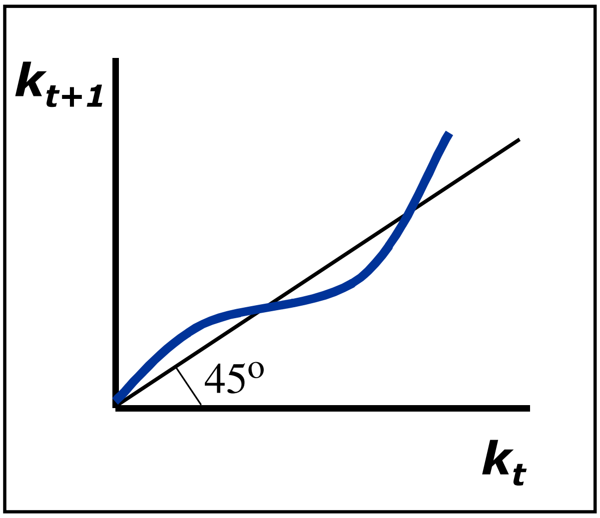
1. Suppose, as in the Diamond model, that N_t individuals are born in each period t. Each individual/consumer lives for two periods only. Utility is given by $\ln(C_{1t}) + \frac{1}{1+\rho}\ln(C_{2t+1})$.
   a. Write down the individual maximization problem
   b. Using the Lagrangean method, find the first-order conditions of the maximisation problem
   c. Derive the marginal propensity to consume and the savings rate of this economy.

2. Suppose that the relationship between k_{t+1} and k_t (capital per unit of effective labour) is given by:
   a) $k_{t+1} = D \cdot k_t \qquad (0 < D < 1)$
   b) $k_{t+1} = D \cdot k_t^{\alpha} \qquad (0 < \alpha < 1)$
   c) $k_{t+1} = D \cdot k_t + B \qquad (0 < D < 1, B > 0)$

   Illustrate the relationship between k_{t+1} and k_t in a diagram and compute the steady state in each case.

3. Suppose that the relationship between k_{t+1} and k_t can be illustrated by the following diagram:

14.



How many steady states can you identify? Which steady state(s) are stable, and which one(s) are unstable?

4. Consider a Diamond economy with Cobb-Douglas production and logarithmic utility. The growth rate of population is equal to 5% and the growth rate of technology is equal to 6%. The rate of time discount ρ is equal to 10% and the capital share α is equal to 0.3.
   a. Compute the steady-state capital per unit of effective labour and the savings rate of this economy
   b. Assume that the rate of time discount ρ decreases to 5%. What is the impact on the economy? Illustrate your answer by means of the appropriate diagram. Compute the steady-state capital per unit of effective labour and the savings rate.
5. “In the Diamond model, a stable equilibrium may not be possible or it may depend on initial conditions”. Discuss.

16 The basic OLG model with and without money

The basic model

Suppose, as in the Diamond model, that N_t individuals are born in each period t. Each individual/consumer lives for two periods only. Population (or labour force) grows at rate g_N, so $N_t = (1 + g_N) N_{t-1}$. For simplicity, let utility be logarithmic with no discounting:

$$U_t = \ln(C_{1t}) + \ln(C_{2t+1}) \tag{7}$$

The production side of the economy is simpler than in the Diamond model. Each individual born at time t is endowed with A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units of the good in the following period.

What is the decentralised equilibrium of this economy?

In the decentralised equilibrium, there will be no trade among members of different generations. Even if the young would like to trade goods this period for goods next period, the only people around to trade with are the old. Unfortunately, the old will be dead next period, and thus in no position to complete the trade.

Therefore, denoting with F_t the amount stored, we obtain the individual budget constraints at dates t and $t+1$:

$$C_{1t} = A - F_t$$
$$C_{2t+1} = x \cdot F_t$$

We can substitute out and combine the two constraints, in this way we obtain the intertemporal budget constraint:

$$C_{1t} + \frac{C_{2t+1}}{x} = A \tag{8}$$

The individual problem is to maximise utility (7), subject to the constraint (8). Using the Lagrangean method, we can obtain the first-order condition of the maximisation problem:

$$C_{2t+1} = x \cdot C_{1t}$$

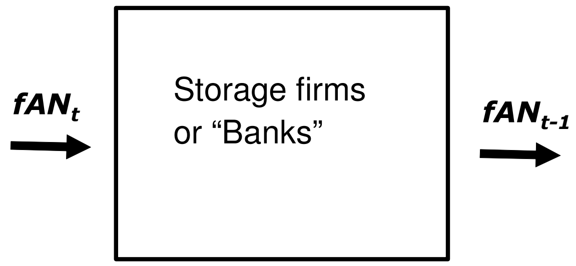
If we substitute the first-order condition into the budget constraint we obtain the solution of the model:

$$C_{1t} = \frac{1}{2}A$$
$$C_{2t+1} = x \cdot \frac{1}{2}A$$

Therefore, $F_t = \frac{1}{2}A$: each individual consumes half of her endowment and stores the other half. The fraction of the endowment that is stored is $f = \frac{F_t}{A} = \frac{1}{2}$. It is constant over time.

Notice that the amount that is stored at each date t is $F_t N_t = fAN_t$, cannot exceed the amount that is reclaimed $xF_{t-1}N_{t-1} = xfAN_{t-1}$, because storage firms will want to make a profit:

15.



$$fAN_t - xfAN_{t-1} > 0$$

$$fAN_{t-1}\left(\frac{N_t}{N_{t-1}} - x\right) = fAN_{t-1}\left(1 + g_N - x\right)$$

Therefore $1 + g_N > x$.

Is this economy efficient?

In each period, total consumption is equal to:

$$C_t = C_{1t} \cdot N_t + C_{2t} \cdot N_{t-1}$$

Each young person consumes the fraction of her endowment that she does not store, $(1 - f)\,A$, and each old person gets to consume the gross return on the fraction of her endowment that she stored fxA. Thus:

$$C_t = (1 - f)\,A \cdot N_t + fxA \cdot N_{t-1}$$

$$\frac{C_t}{N_t} = (1 - f)\,A + fxA \cdot \frac{N_{t-1}}{N_t}$$

$$\frac{C_t}{N_t} = (1-f)\,A + fxA \cdot \frac{1}{1+g_N}$$
$$\frac{C_t}{N_t} = \left[1 - f + f \cdot \frac{x}{1+g_N}\right] A$$
$$\frac{C_t}{N_t} = A - f\left(1 - \frac{x}{1+g_N}\right) A$$

Notice that consumption per capita is maximised when $x = 1 + g_N$. So, if $x < 1 + g_N$ this economy could become more efficient with a "social planner". A social planner could take A units from each young person and give $(1+g_N)\,A$ units to each old person. With $x < 1 + g_N$, this gives a better return than storage.

Monetary model

Now suppose that, instead of storage, individuals can trade in units of a storable, divisible commodity, which we will call "money" and denote with M. Money is not a source of utility.

The price of goods in units of money at time t is P_t , and P_{t+1} in $t+1$. Thus the individual can sell units of endowment for units of money and then use that money to buy units of the next generation's endowment.

The individual budget constraints at dates t and $t+1$ now become:

$$P_t C_{1t} = P_t A - M_t$$

$$P_{t+1} C_{2t+1} = M_t$$

If we combine the two constraints we get the intertemporal budget constraint:

$$P_t C_{1t} + P_{t+1} C_{2t+1} = P_t A \tag{9}$$

$$C_{1t} + \frac{C_{2t+1}}{\frac{P_t}{P_{t+1}}} = A$$

Notice that the rate of return on money is $\frac{P_t}{P_{t+1}}$, since the individual can sell one unit of consumption in period t and get P_t units of money. In period $t+1$, one unit of consumption costs P_{t+1} units of money and thus one unit of money will buy $\frac{1}{P_{t+1}}$ units of consumption. Thus the individual's P_t units of money will buy $\frac{P_t}{P_{t+1}}$ units of consumption in period $t+1$.

The individual problem is to maximise (7), subject to the constraint (9). We can obtain the first-order condition:

$$P_t C_{1t} = P_{t+1} C_{2t+1}$$

And by substituting the first-order condition into the budget constraint we obtain the solution of the model:

$$C_{1t} = \frac{1}{2} A$$

$$C_{2t+1} = \frac{P_t}{P_{t+1}} \frac{1}{2} A$$

Thus, in period t the individual consumes half of her endowment and then sells the rest for money:

$$M_t = \frac{1}{2} P_t A$$

Equilibrium

Equilibrium requires that aggregate money demand equal aggregate money supply. We can derive expressions for both money demand and money supply in period $t+1$:

Aggregate money demand = $N_{t+1}\left(\frac{1}{2}P_{t+1}A\right)$

Aggregate money supply = $N_t\left(P_{t+1}C_{2t+1}\right) = N_t P_{t+1}\frac{P_t}{P_{t+1}}\frac{1}{2}A = N_t P_t \frac{1}{2}A$

Equilibrium:

$$N_{t+1}\frac{1}{2}P_{t+1}A = N_t P_t \frac{1}{2}A$$

Thus:

$$\frac{P_t}{P_{t+1}} = \frac{N_{t+1}}{N_t} = 1 + g_N$$

This shows that if money is introduced, the economy becomes dynamically efficient, since the return on money is equal to $1+g_N$. As shown above, this ensures that consumption per capita is maximised in each period.

Revision questions

1. Explain why money enables members of different generations to trade with each other. Using a simple OLG model, demonstrate that consumption per capita is maximised in a monetary economy, but not in an economy without money.